Exam Analysis on Manifolds

WIANVAR-07.2018-2019.1B

January 29, 2019

This exam consists of **four** assignments. You get 10 points for free.

Assignment 1. (20 pt.)
Let $M = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$, and let $i : M \to \mathbb{R}^3$ be the inclusion map. We turn M into a topological space by endowing it with the subspace topology inherited from $\mathbb{R}^3$. You may assume that this subspace topology of M is second-countable, and that M with this topology is a Hausdorff space.

1. Construct an atlas on M such that:

   (i) (7 pt.) M becomes a two-dimensional C^∞-manifold;
   (ii) (5 pt.) the inclusion map i is a C^∞-map.

2. (8 pt.) Let σ be the one-form on $\mathbb{R}^3$ given by $\sigma = x\,dx + y\,dy$.
   Prove that $i^*\sigma = 0$.

Solution.
1.(i) Note that M is the Cartesian product of the unit circle in $\mathbb{R}^2$ and the real line $\mathbb{R}$. We will construct an atlas on M based on this observation. To this end, let $U_1 = \{(\cos u, \sin u, v) \mid 0 < u < 2\pi, v \in \mathbb{R}\}$ and let $U_2 = \{(\cos u, \sin u, v) \mid -\pi < u < \pi, v \in \mathbb{R}\}$. Let $\varphi_i : U_i \to \mathbb{R}^2$ be defined by

$$\varphi_i(\cos u, \sin u, v) = (u, v).$$

Then φ_i is a homeomorphism onto its image V_i, where $V_1 = (0, 2\pi) \times \mathbb{R}$ and $V_2 = (-\pi, \pi) \times \mathbb{R}$. Note that $V_1 \cap V_2 = ((0, \pi) \times \mathbb{R}) \cup ((\pi, 2\pi) \times \mathbb{R})$. Furthermore,

$$\varphi_2 \circ \varphi_1^{-1}(u, v) = \begin{cases} u, & \text{if } (u, v) \in (0, \pi) \times \mathbb{R}, \\ u - 2\pi, & \text{if } (u, v) \in (\pi, 2\pi) \times \mathbb{R}. \end{cases}$$

Hence $\varphi_2 \circ \varphi_1^{-1}$ is a diffeomorphism. Therefore, $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is a C^∞-atlas on M, turning M into a two-dimensional manifold.

1.(ii). To prove that the inclusion map is C^∞, observe that $\mathrm{id} \circ i \circ \varphi_i^{-1} : V_i \to \mathbb{R}^3$ is equal to $g_i : V_i \to \mathbb{R}^3$, defined by $g_i(u, v) = (\cos u, \sin u, v)$. Note that on the manifold $\mathbb{R}^3$ we use the atlas $(\mathbb{R}^3, \mathrm{id})$. Also note that φ_i^{-1} and g_i only differ in the sense that they have different ranges. Since g_i is C^∞, the claim follows.

2. We shall prove that both local representatives $(\varphi_i^{-1})^*(i^*\sigma)$, $i = 1, 2$, are zero. If f is either of the maps φ_i^{-1}, then $f^*(i^*\sigma) = (i \circ f)^*(\sigma)$, and $(i \circ f)(u, v) = (\cos u, \sin u, v)$. Hence, $f^*(i^*\sigma) = \cos u\, d(\cos u) + \sin u\, d(\sin u) = -\cos u \sin u\, du + \sin u \cos u\, du = 0$.

Assignment 2. (25 pt.)
Let X be a vector field on $\mathbb{R}^3$, and let $\Omega = dx_1 \wedge dx_2 \wedge dx_3$, a 3-form on $\mathbb{R}^3$. Recall that, for a k-form ω, $k > 0$, the $(k-1)$-form $\iota_X\omega$ is defined by

$$\iota_X\omega(X_1, \dots, X_{k-1}) = \omega(X, X_1, \dots, X_{k-1}).$$

1. (8 pt.) Let $\sigma_1, \dots, \sigma_3$ be one-forms on $\mathbb{R}^3$. Show that

$$\iota_X(\sigma_1 \wedge \sigma_2 \wedge \sigma_3) = \sigma_1(X)\, \sigma_2 \wedge \sigma_3 - \sigma_2(X)\, \sigma_1 \wedge \sigma_3 + \sigma_3(X)\, \sigma_1 \wedge \sigma_2.$$

2. (7 pt.) Prove that for every 2-form ω on $\mathbb{R}^3$ there is a vector field Z on $\mathbb{R}^3$ such that

$$\omega = \iota_Z\Omega.$$

3. (10 pt.) Prove that $d(\iota_X\Omega) = 0$ if and only if there is a vector field Y on $\mathbb{R}^3$ such that $X = \nabla \times Y$ (the rotation of Y, also known as the curl of Y).

Solution.
1. Let X_1 and X_2 be arbitrary vector fields on $\mathbb{R}^3$, then

$$\iota_X(\sigma_1 \wedge \sigma_2 \wedge \sigma_3)(X_1, X_2) = \begin{vmatrix} \sigma_1(X) & \sigma_1(X_1) & \sigma_1(X_2) \\ \sigma_2(X) & \sigma_2(X_1) & \sigma_2(X_2) \\ \sigma_3(X) & \sigma_3(X_1) & \sigma_3(X_2) \end{vmatrix}$$

$$= \sigma_1(X) \begin{vmatrix} \sigma_2(X_1) & \sigma_2(X_2) \\ \sigma_2(X_1) & \sigma_3(X_2) \end{vmatrix} - \sigma_2(X) \begin{vmatrix} \sigma_1(X_1) & \sigma_1(X_2) \\ \sigma_3(X_1) & \sigma_3(X_2) \end{vmatrix}$$

$$+ \sigma_3(X) \begin{vmatrix} \sigma_1(X_1) & \sigma_1(X_2) \\ \sigma_2(X_1) & \sigma_2(X_2) \end{vmatrix}$$

$$= \sigma_1(X)\, \sigma_2 \wedge \sigma_3(X_1, X_2) - \sigma_2(X)\, \sigma_1 \wedge \sigma_3(X_1, X_2) + \sigma_3(X)\, \sigma_1 \wedge \sigma_2(X_1, X_2)$$

$$= (\sigma_1(X)\, \sigma_2 \wedge \sigma_3 - \sigma_2(X)\, \sigma_1 \wedge \sigma_3 + \sigma_3(X)\, \sigma_1 \wedge \sigma_2)\, (X_1, X_2).$$

2. Let $\omega = f_1\, dx^2 \wedge dx^3 - f_2\, dx^1 \wedge dx^3 + f_3\, dx^1 \wedge dx^2$, Taking $\sigma_i = dx^i$ in part 1, we see that $\omega = \iota_X\Omega$ if we take X such that $dx^i(X) = f_i$, i.e., if $X = \sum_{i=1}^3 f_i \frac{\partial}{\partial x^i}$.

3. Let $Y = f_1 \frac{\partial}{\partial x^1} + f_2 \frac{\partial}{\partial x^2} + f_3 \frac{\partial}{\partial x^3}$. Then

$$\nabla \times Y = (\frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3}) \frac{\partial}{\partial x^1} - (\frac{\partial f_3}{\partial x^1} - \frac{\partial f_1}{\partial x^3}) \frac{\partial}{\partial x^2} + (\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2}) \frac{\partial}{\partial x^3}.$$

Therefore,

$$\iota_{\nabla\times Y}\Omega = (\frac{\partial f_3}{\partial x^2} - \frac{\partial f_2}{\partial x^3})\, dx^2 \wedge dx^3 - (\frac{\partial f_3}{\partial x^1} - \frac{\partial f_1}{\partial x^3})\, dx^1 \wedge dx^3 + (\frac{\partial f_2}{\partial x^1} - \frac{\partial f_1}{\partial x^2})\, dx^1 \wedge dx^2.$$

In other words, $\iota_{\nabla\times Y}\Omega = d\eta$, for $\eta = f_1\, dx^1 + f_2\, dx^2 + f_3\, dx^3$. This shows that $d\iota_X\Omega = 0$ for $X = \nabla \times Y$.

Conversely, if $d\iota_X\Omega = 0$, then by Poincaré's Lemma there is a one-form η on $\mathbb{R}^3$ such that $\iota_X\Omega = d\eta$. Let $\eta = f_1\,dx^1 + f_2\,dx^2 + f_3\,dx^3$, then $X = \nabla \times Y$ for $Y = f_1\frac{\partial}{\partial x^1} + f_2\frac{\partial}{\partial x^2} + f_3\frac{\partial}{\partial x^3}$.

Assignments 3 and 4 on next page

Assignment 3. (20 pt.)
In this exercise M and N are C^∞-manifolds, and $F : N \to M$ is a C^∞-map. For $p \in N$ the map $F^* : C^\infty_{F(p)}(N) \to C^\infty_p(M)$ is the usual pullback given by $F^*(g) = g \circ F$.

1. (5 pt.) Assume that F^* is surjective. Prove that $F_{*,p} : T_pN \to T_{F(p)}M$ is injective.

Let $f : \mathbb{R}^{n+1} \to \mathbb{R}$ be a C^∞-function for which 0 is a regular value. Therefore, $N = F^{-1}(0)$ is a C^∞-submanifold of $\mathbb{R}^{n+1}$.

Recall that a function $g : U \to \mathbb{R}$, defined on an open subset U of N, is C^∞ if for every $p \in U$ there is a neighborhood V of p in $\mathbb{R}^{n+1}$ and a C^∞-function $\tilde{g} : V \to \mathbb{R}$ such that $g = \tilde{g}$ on $V \cap U$. Let $i : N \to \mathbb{R}^{n+1}$ be the inclusion map.

2. (5 pt.) Prove that $i^* : C^\infty_p(\mathbb{R}^{n+1}) \to C^\infty_p(N)$ is surjective for $p \in N$.

3. (3 pt.) Prove that $i_{*,p} : T_p(N) \to T_p(\mathbb{R}^{n+1})$ is injective for $p \in N$.

4. (7 pt.) Consider the map $f_{*,p} : T_p\mathbb{R}^{n+1} \to T_0\mathbb{R}$ for $p \in N$.
   Prove that $i_{*,p}(T_p(N)) = \ker f_{*,p}$.

Solution.
1. Since F_* is a linear map, we have to prove that $\ker F_* = \{0\}$. Let $X_p \in T_pN$ and assume $F_*(X_p) = 0$. We have to show that $X_p = 0$, , i.e., that $X_p(f) = 0$ for all $f \in C^\infty_p(N)$. So let $f \in C^\infty_p(N)$, then there is a $g \in C^\infty_{F(p)}(M)$ such that $f = F \circ g$. Then, for all $f \in C^\infty_p(N)$:

$$X_p(f) = X_p(F \circ g) = (F_*(X_p))(g) = 0.$$

Therefore, $X_p = 0$, so F_* is injective.

2. Let a germ in $C^\infty_p(N)$ be represented by a C^∞ function $g : U \to \mathbb{R}$, where U is a neighborhood of p in N. Then there is a C^∞ function $\tilde{g} : V \to \mathbb{R}$ defined on a neighborhood V in $\mathbb{R}^{n+1}$, such that $g = \tilde{g}$ on $U \cap V$. In other words, $g = \tilde{g} \circ i$ on the neighborhood $U \cap V$ of p in N, so $[g] = [\tilde{g} \circ i] = i^*[\tilde{g}]$.

3. This result is a straightforward consequence of Parts 1 and 2 of this exercise.

4. Note that $\ker f_{*,p}$ is an n-dimensional subspace of $T_p\mathbb{R}^{n+1}$. Part 3 implies that $i_{*,p}(T_pN)$ is also an n-dimensional subspace of $T_p\mathbb{R}^{n+1}$, so it is sufficient to prove that $i_{*,p}(T_pN) \subset \ker f_{*,p}$. This inclusion follows from the fact that $f \circ i : N \to \mathbb{R}$ is the zero-function, so $f_{*,p} \circ i_{*,p} = (f \circ i)_{*,p} = 0$. $\square$

Assignment 4. (25 pt.)
Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞-function such that $f|_{\mathbb{S}^1} = 0$. Here $\mathbb{S}^1$ is the unit circle in $\mathbb{R}^2$, the boundary of the closed unit disc $\mathbb{B}^2$ in $\mathbb{R}^2$. The goal of this assignment is to prove that if f is harmonic, i.e., if the Laplacian of f is zero on $\mathbb{B}^2$, then $f = 0$ on $\mathbb{B}^2$.

As usual, the Laplacian Δf of f is given by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

1. (8 pt.) Determine a one-form ω on $\mathbb{R}^2$ such that $d\omega = (\Delta f) dx \wedge dy$.
2. (9 pt.) Prove that

$$\int_{\mathbb{B}^2} \left(f\,\Delta f + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2\right) dx \wedge dy = 0.$$

Hint: prove that the integrand is equal to $d\psi$ for $\psi = f\omega$ and ω as in Part 1 of this assignment.

3. (8 pt.) Prove: If $\Delta f = 0$ on $\mathbb{B}^2$, then $f|_{\mathbb{B}^2} = 0$.

Solution.
1. Let $\omega = a dx + b dy$ for C^∞-functions $a, b : \mathbb{R}^2 \to \mathbb{R}$. Then

$$d\omega = (-\frac{\partial a}{\partial y} + \frac{\partial b}{\partial x}) dx \wedge dy,$$

so we have to determine a and b such that

$$-\frac{\partial a}{\partial y} + \frac{\partial b}{\partial x} = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}.$$

A solution is $a = -\frac{\partial f}{\partial y}$ and $b = \frac{\partial f}{\partial x}$, i.e.,

$$\omega = -\frac{\partial f}{\partial y} dx + \frac{\partial f}{\partial x} dy.$$

2. A straightforward computation shows that the integrand is equal to $d\psi$, with $\psi = f\omega$. Now apply Stokes's Theorem:

$$\int_{\mathbb{B}^2} \left(f\,\Delta f + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2\right) dx \wedge dy = \int_{\mathbb{B}^2} d\psi = \int_{\mathbb{S}^1} \psi = \int_{\mathbb{S}^1} f\omega = 0. \quad (1)$$

3. If $\Delta f = 0$ on $\mathbb{B}^2$, we get from (1):

$$\int_{\mathbb{B}^2} \left((\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2\right) dx \wedge dy = 0.$$

Since the integrand is non-negative, it has to be identically zero. This implies that

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

on $\mathbb{B}^2$. Hence, f is constant on $\mathbb{B}^2$. Since $f|_{\mathbb{S}^1} = 0$, this constant is equal to zero. □